Math 2400 Section 003 – Calculus III – Spring 2012

Chapter 12 Sections 5 & 6 Notes

You can find the definition of level set, limit, and continuity in your text. I'll add some understanding to those definitions here. Also, we used a sage worksheet in class to introduce the various shapes that will be important to this course. You can find that worksheet at `http://sage.colorado.edu/home/pub/2`

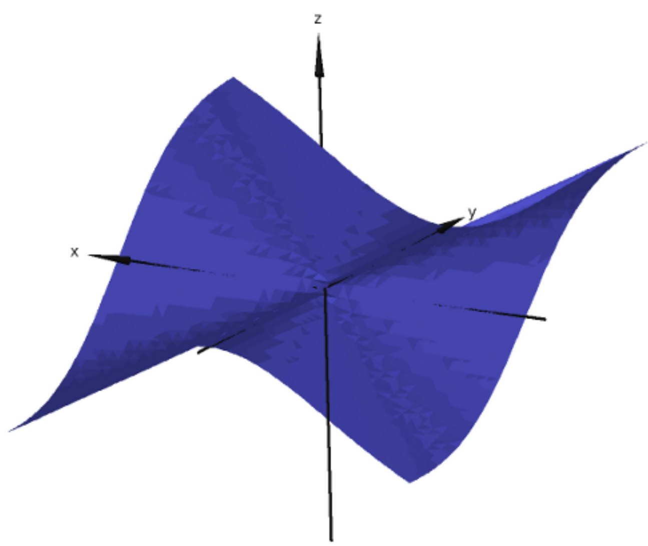
Limit Examples

Example 1 The function $f(x,y) = \dfrac{x^2y}{x^2+y^2}$ is clearly not continuous at the origin since it is not defined there. But, does the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exist?

Solution: By the definition of limit, we want some value L such that $f(x,y)$ is very close to L as (x,y) is substantially close to $(0,0)$. In other words, if we can figure out a way to make $f(x,y)$ approach some value L reliably as we approach $(0,0)$, we will probably be done. Looking at the graph of this function, we see that the value of the function appears to approach 0 as (x,y) approaches $(0,0)$. We saw the graph in class. It looks like this:



Thus, it appears that the value of the function is approaching zero near the origin. So, let's work under the assumption that maybe $L = 0$. How much different is $f(x,y)$ from L? Well, if we can bound the difference between them and show that this difference goes to zero the closer we get to the origin, we'll be done. Here's a convincing argument that seemingly comes out of nowhere: If $L = 0$ then

$$|f(x,y) - L| = |f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right| |y| \le |y| \le \sqrt{x^2+y^2}.$$

If you have questions on this identity, please ask. It's not at all immediately obvious. (Half of advanced math is full of identities like this. They look completely awkward until you udnerstand them.) But, it does limit the difference between the value of the function and the value of L as we get closer to the origin. (Specifically, it shows that the difference between $f(x,y)$ and L is always bounded above by the Euclidean distance from $(0,0)$ to (x,y). Thus, the limit exists and is equal to zero.

Example 2 The function $g(x,y) = \dfrac{x^2}{x^2+y^2}$ is also clearly not continuous at the origin (since it is not defined there). Does the function have a limit that exists at the origin?

Solution: If we approach the point $(0,0)$ along the x-axis, where $y = 0$, and make sure to consider only x values that are nonzero (the function is not defined at the origin) then we find that the function always has the value

$$g(x,0) = \frac{x^2}{x^2+0} = \frac{x^2}{x^2} = 1.$$

On the other hand, if we approach the origin along the y-axis, where $x = 0$, and again make sure to only consider points where $y \neq 0$, then we always have

$$g(0,y) = \frac{0}{0+y^2} = 0.$$

This is a problem, since we can approach the origin along two different paths (either axes in this case) and have the function approach a different value along each path. Thus, the limit does not exist at the origin.

Example 3 Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ exist?

Solution: As before, the best way to show that a limit doesn't exist is generally to approach the point in question along various paths. If one approaches the origin along the x-axis, then we always have

$$\frac{x+y}{x-y} = \frac{x}{x} = 1.$$

If however, one decides to approach along the line $y = 2x$, then we would always have

$$\frac{x+y}{x-y} = \frac{x+2x}{x-2x} = \frac{3x}{-x} = -3.$$

Thus, the limit does not exist.