Math 2400 Section 003 – Calculus III – Spring 2012

Chapter 13 Section 3 Notes

Algebraic definition of the dot product: Given two vectors $\vec{v}$ and $\vec{w}$ of dimension n,

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

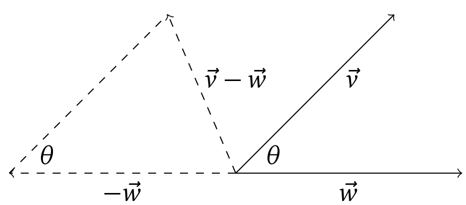
Geometric definition of the dot product: Given two vectors $\vec{v}$ and $\vec{w}$ with the (smallest) angle between them being θ,

$$\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}|| \cos\theta.$$

Proof of Dot Product Equality

Theorem: The two definitions of the dot product yield the same object.

Proof: To prove that the two definitions of the dot product are in fact the same, consider the following diagram.



In this situation, the law of cosines states that $||\vec{v} - \vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||\,||\vec{w}|| \cos\theta$. If we expand the left side of this equation, we get (from the distance formula)

$$||\vec{v} - \vec{w}||^2 = \left(\sqrt{(v_1 - w_1)^2 + \cdots + (v_n - w_n)^2}\right)^2 = \sum_{i=1}^{n} (v_i - w_i)^2 = \sum_{i=1}^{n} (v_i^2 - 2v_i w_i + w_i^2).$$

We can separate the sum and we are left with the left hand side of the given equation equal to

$$\sum_{i=1}^{n} v_i^2 - 2\sum_{i=1}^{n} v_i w_i + \sum_{i=1}^{n} w_i^2$$

If we expand parts of the right hand side using the same idea, we have

$$||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||\,||\vec{w}|| \cos\theta = \sum_{i=1}^{n} v_i^2 + \sum_{i=1}^{n} w_i^2 - 2||\vec{v}||\,||\vec{w}|| \cos\theta.$$

So, after transforming the left and right sides separately, we now have that

$$\sum_{i=1}^{n} v_i^2 - 2\sum_{i=1}^{n} v_i w_i + \sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} v_i^2 + \sum_{i=1}^{n} w_i^2 - 2||\vec{v}||\,||\vec{w}|| \cos\theta.$$

Notice that this simplifies to

$$\sum_{i=1}^{n} v_i w_i = ||\vec{v}||\,||\vec{w}|| \cos\theta,$$

and we're done. (The sum on the left is the algebraic version and the right side is the geometric version.) □

The Equation of a Plane

We discussed the definitions of "perpendicular" (also: "orthogonal") in class. This lead to the definition of a normal vector. Those are all very important and you should become familiar with them.

If you are given a point $P_0 = (x_0, y_0, z_0)$ on a plane and a normal vector $\vec{n}$ to the plane, you can construct the equation of the plane very quickly as

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0.$$

Someone asked in class how this relates to the equation of a plane we already know from Chapter 12:

$$z = \frac{\Delta z}{\Delta x}(x - x_0) + \frac{\Delta z}{\Delta y}(y - y_0) + z_0.$$

Here's how. If we solve the top equation for z, we have

$$z = -\frac{n_1}{n_3}(x - x_0) - \frac{n_2}{n_3}(y - y_0) + z_0.$$

The key point is that

$$\frac{\Delta z}{\Delta x} = -\frac{n_1}{n_3} \qquad \frac{\Delta z}{\Delta y} = -\frac{n_2}{n_3}.$$

To see this, you should notice that the left side of each of these equalities is viewed as being on the plane. The right sides are calculated along the normal vector. The plane and the normal vector are orthogonal. How do you calculate slopes of orthogonal lines? Remember from algebra that perpendicular lines have negative reciprocal slopes. To be more specific, we have

$$\frac{\Delta z}{\Delta x} = -\frac{n_1}{n_3} = -\left(\frac{n_3}{n_1}\right)^{-1}$$

where you should realize that n_3 is simply the change in z along the normal vector and n_1 is the change in x along the normal vector. So, these really are the same equations of the plane, just written in two drastically different ways.

Examples

Example 1: Find a normal vector to the plane $z + 2(x - 2) = 8(2 - y)$.

Solution: If we can place this equation in the form $n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0) = 0$, and determine what the components n_i must be, we'll be done. In fact, as showed in class, the given plane can be pushed around algebraically until we get

$$2(x - 2) + 8(y - 2) + 1(z - 0) = 0.$$

Thus, a normal vector is $\vec{n} = 2\vec{i} + 8\vec{j} + \vec{k}$.

Example 2: Find a vector parallel to the plane in the previous example.

Solution: Any vector parallel to the plane must also be orthogonal to any normal vector for the plane. So, our goal is simply to create any vector that is orthogonal to the normal vector we found in the previous example. We use the algebraic version of the dot product to do this. (There are infinitely many solutions.) For instance, we know that if $\vec{v}$ is our parallel vector, then

$$v_1 n_1 + v_2 n_2 + v_3 n_3 = 2v_1 + 8v_2 + v_3 = 0$$

by the definition of orthogonality. We could use $v_1 = -8$, $v_2 = 2$, and $v_3 = 0$.

Example 3: Find the angle between $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$ and $\vec{w} = -\vec{i} + 3\vec{k}$.

Solution: Both versions of the dot product will be used here. Namely, the algebraic version is on the left and the geometric version is on the right. We have

$$(1)(-1) + (2)(0) + (1)(3) = ||\vec{v}||\,||\vec{w}|| \cos\theta.$$

Calculating the norm of the vectors and simplifying gives us

$$2 = \sqrt{60}\cos\theta = 2\sqrt{15}\cos\theta.$$

Thus, we have $\cos\theta = 1/\sqrt{15}$, or $\theta = \cos^{-1}(1/\sqrt{15})$.