Calculus 3 – Spring 2012

Written Homework #8

Due 3/23/2012

1. The integral in this problem is one of the most important in all of mathematics, but it has a non-trivial solution. The purpose of this problem is to guide you through that solution. We wish to compute
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$
   (a) We'll calculate this integral by working with an entirely different integral. Consider the double integral
   $$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$
   Show (without evaluating the integral, and with staying in Cartesian coordinates) that this new integral satisfies the equality
   $$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2.$$
   (b) Rewrite the integral
   $$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$$
   in polar coordinates and solve for a numerical value. *Hints: You will need to use substitution and evaluate an improper integral. Make sure to change your limits of integration to reflect your substitution. The numerical value you obtain should be a nice round number.*

   (c) Putting (a) and (b) together, can you determine the value of the original integral?

2. Find the volume of an ice cream cone bounded above by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and bounded below by the cone $z = \sqrt{x^2 + y^2}$.

3. Evaluate the integral
$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} \frac{1}{(x^2+y^2+z^2)^{1/2}}\,dy\,dz\,dx.$$

4. Electric charge is distributed throughout 3-space with density proportional to the distance from the xy-plane. Show that the total charge inside a cylinder of radius R and height h, sitting on the xy-plane and centered along the z-axis, is proportional to R^2h^2.

5. For the change of variables $x = 3s - 4t$ and $y = 5s + 2t$, show that
$$\frac{\partial(x,y)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)} = 1.$$

6. Use the change of variables $s = 1/r$ and $t = \theta$ to transform and evaluate the polar coordinate integral
$$\int_R (1/r^3)\,r\,dr\,d\theta$$
over the infinite region R where $r \geq 1$ and $0 \leq \theta \leq 2\pi$ to an iterated integral over a finite region.