Math 2400 Section 003 – Calculus III – Spring 2011

Integration by Parts Examples

Classic Examples

1. $\int xe^x \, dx$

Solution: We select u and v' and then u' and v follow.

$$u = x \qquad v' = e^x$$
$$u' = 1 \qquad v = e^x$$

$$\begin{aligned} \int xe^x \, dx &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C \end{aligned}$$

2. $\int x \sin x \, dx$

Solutions:

$$u = x \qquad v' = \sin x$$
$$u' = 1 \qquad v = -\cos x$$

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

Integration By Parts Multiple Times

1. $\int x^2 \sin x \, dx$

Solution: What happens here is that the first application of integration by parts results in another integral that is best approached by using integration by parts.

$$u_1 = x^2 \qquad v_1' = \sin x$$
$$u_1' = 2x \qquad v_1 = -\cos x$$

$$\begin{aligned} \int x^2 \sin x \, dx &= x^2 \cos x + \int 2x \cos x \, dx \\ &= x^2 \cos x + 2 \int x \cos x \, dx \end{aligned}$$

$$u_2 = x \qquad v_2' = \cos x$$
$$u_2' = 1 \qquad v_2 = \sin x$$

$$\begin{aligned} &= x^2 \cos x + 2 \left(x \sin x - \int \sin x \, dx \right) \\ &= x^2 \cos x + 2x \sin x + 2 \cos x + C \end{aligned}$$

2. $\int x^2 e^x \, dx$

Solution: Again, we use integration by parts twice. The subscripts on the variables below tell us during which application of integration by parts those variables were used.

$$u_1 = x^2 \qquad v_1' = e^x$$
$$u_1' = 2x \qquad v_1 = e^x$$
$$u_2 = 2x \qquad v_2' = e^x$$
$$u_2' = 2 \qquad v_2 = e^x$$

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - \int 2xe^x \, dx \\ &= x^2 e^x - \left(2xe^x - \int 2e^x \, dx\right) \\ &= x^2 e^x - 2xe^x - 2e^x + C.\end{aligned}$$

Definite Integral Examples

1. $\displaystyle\int_1^5 \ln x \, dx$

   This is also a good example of how you can sometimes use integration by parts with one of the functions being trivial (i.e., $v' = 1$).

   Solution: When we use integration by parts, we simply keep in mind that we're evaluating the original integral from 1 to 5. That means that we must evaluate BOTH terms (the $x \ln x$ and the integral) given by integration by parts. Also, it helps to remember here that $\ln 1 = 0$.

$$u = \ln x \qquad v' = 1$$
$$u' = \frac{1}{x} \qquad v = x$$

$$\begin{aligned}\int_1^5 \ln x \, dx &= x \ln x \Big|_1^5 - \int_1^5 dx \\ &= 5 \ln 5 - \ln 1 - 4 \\ &= 5 \ln 5 - 4\end{aligned}$$

More Complicated Examples

1. $\displaystyle\int \cos^2(3\alpha + 1) \, d\alpha$

   Solutions: We need to use integration by parts, plus a trig identity, plus substitution in order to compute this integral.

$$u = \cos(3\alpha + 1) \qquad v' = \cos(3\alpha + 1)$$
$$u' = -3\sin(3\alpha + 1) \qquad v = \frac{1}{3}\sin(3\alpha + 1)$$

$$\begin{aligned}\int \cos^2(3\alpha + 1) \, d\alpha &= \int \cos(3\alpha + 1)\cos(3\alpha + 1) \, d\alpha \\ &= \frac{1}{3}\cos(3\alpha + 1)\sin(3\alpha + 1) + \int \sin^2(3\alpha + 1) \, d\alpha\end{aligned}$$

By using the trig identity $\cos 2\alpha = 1 - 2\sin^2\alpha$ and solving for $\sin^2\alpha$ we can write the above as

$$= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \int \frac{1}{2} - \frac{1}{2}\cos(6\alpha+2)\,d\alpha$$

Using the substitution $w = 6\alpha + 2$ (and so $dw = 6\,d\alpha$) this then becomes

$$\begin{aligned}
&= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \frac{1}{4}x^2 - \frac{1}{12}\int \cos w\,dw \\
&= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \frac{1}{4}x^2 - \frac{1}{12}\sin w + C \\
&= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \frac{1}{4}x^2 - \frac{1}{12}\sin(6\alpha+2) + C
\end{aligned}$$

Recursive Examples

1. $\int e^x \sin x\,dx$

   Solution: In this case, neither e^x or $\sin x$ will become simpler when differentiated, and so at the start it doesn't seem like integration by parts can help. But, something interesting happens.

$$\begin{aligned}
u_1 &= e^x & v_1' &= \sin x \\
u_1' &= e^x & v_1 &= -\cos x \\
u_2 &= e^x & v_2' &= \cos x \\
u_2' &= e^x & v_2 &= \sin x
\end{aligned}$$

$$\begin{aligned}
\int e^x \sin x\,dx &= -e^x\cos x + \int e^x \cos x\,dx \\
&= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx
\end{aligned}$$

   The main thing to notice here is that the integral we are trying to find is on both sides of the equation, and since the right hand side version is negative, we add it to both sides. We get

$$2\int e^x \sin x\,dx = -e^x\cos x + e^x\sin x$$

   and dividing both sides by 2 gives

$$\int e^x \sin x\,dx = \frac{1}{2}\left(-e^x\cos x + e^x \sin x\right) + C.$$

2. $\int \cos^2\theta\,d\theta$

   Solution: There are two ways to consider this problem. One is to use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

   and substitution. The other is to use the Pythagorean identity

$$\cos^2\theta = 1 - \sin^2\theta$$

and integration by parts. The latter is done here.

$$u = \cos\theta \qquad v' = \cos\theta$$
$$u' = -\sin\theta \qquad v = \sin\theta$$

$$\begin{aligned}\int \cos^2\theta\, d\theta &= \cos\theta\sin\theta + \int \sin^2\theta\, d\theta\\ &= \cos\theta\sin\theta + \int 1-\cos^2\theta\, d\theta\\ &= \cos\theta\sin\theta + \int 1\, d\theta - \int \cos^2\theta\, d\theta\end{aligned}$$

Adding the integral of $\cos^2\theta$ to both sides gives

$$\begin{aligned}2\int \cos^2\theta\, d\theta &= \cos\theta\sin\theta + \int 1\, d\theta\\ 2\int \cos^2\theta\, d\theta &= \cos\theta\sin\theta + \theta + C\\ \int \cos^2\theta\, d\theta &= \frac{1}{2}(\cos\theta\sin\theta + \theta) + C.\end{aligned}$$

Reduction Formula Example

9. Prove the formula

$$\int \sin^n\, dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$$

Solution: We did a problem a lot like this in class, with specific numbers. You also had a similar problem on homework. The main idea is to start with one function as $\sin^{n-1}x$ and the other as $\sin x$. Proceed as follows.

$$u = \sin^{n-1}x \qquad v' = \sin x$$
$$u' = (n-1)\sin^{n-2}x\cos x \qquad v = -\cos x.$$

$$\begin{aligned}&\int \sin^n x\, dx\\ &= -\cos x\sin^{n-1}x + \int (n-1)\sin^{n-2}x\cos x\cos x\, dx\\ &= -\cos x\sin^{n-1}x + \int (n-1)\sin^{n-2}x\cos^2 x\, dx\\ &= -\cos x\sin^{n-1}x + \int (n-1)\sin^{n-2}x(1-\sin^2 x)\, dx\\ &= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x - \sin^n x\, dx\\ &= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx - (n-1)\int \sin^n x\, dx\end{aligned}$$

Now move the $(n-1)\int \sin^n x\, dx$ to the left side.

$$\begin{aligned}\int \sin^n x\, dx + (n-1)\int \sin^n x\, dx &= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx\\ n\int \sin^n x\, dx &= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx\\ \int \sin^n x\, dx &= -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\, dx\end{aligned}$$

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