Practice Integrals in Rectangular/Cylindrical/Spherical Coordinates Answers

Setup the following integrals in all reasonable coordinate systems. Compute each quantity in an appropriate coordinate system.

1. The volume the solid bounded by the planes $7x - 8y + 2z = 19$, $5x - y + z = 2$, $y = 3x + 8$, and $x = 4$.

$$\int_{-3}^{4} \int_{-\frac{1}{2}(x+5)}^{3x+8} \int_{2-5x+y}^{\frac{1}{2}(19-7x+8y)} 1\, dz\, dy\, dx = \frac{16807}{8}$$

2. The volume between $z = y^2 + 1$ and $z = 9 - 2x^2 - y^2$.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{y^2+1}^{9-2x^2-y^2} 1\, dz\, dy\, dx = \int_{0}^{2\pi} \int_{0}^{2} \int_{r^2 \sin^2\theta+1}^{9-r^2\left(1+\cos^2\theta\right)} r\, dz\, dr\, d\theta = 16\pi$$

3. The mass of the solid that is bounded by the cone $z = \frac{1}{a}\sqrt{x^2+y^2}$ and the plane $z = b$, and whose density is proportional to the distance from the z-axis.

$$\int_{0}^{2\pi} \int_{0}^{b} \int_{0}^{az} kr \cdot r\, dr\, dz\, d\theta = \int_{0}^{2\pi} \int_{0}^{\tan^{-1} a} \int_{0}^{b \sec\phi} k\rho \sin\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^3 b^4 k\pi}{6}$$

4. The volume inside $x^2 + y^2 = R^2$ in the first octant, and below $z = 3x$.

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{0}^{3x} 1\, dz\, dy\, dx = \int_{0}^{\frac{\pi}{2}} \int_{0}^{R} \int_{0}^{3r\cos\theta} r\, dz\, dr\, d\theta = R^3$$

5. The volume inside $x^2 + y^2 + z^2 = R^2$ and above $z = x^2 + y^2$.

$$\int_{0}^{2\pi} \int_{0}^{\sqrt{\frac{-1+\sqrt{1+4R^2}}{2}}} \int_{r^2}^{\sqrt{R^2-r^2}} r\, dz\, dr\, d\theta = \frac{2}{3}\pi R^3 - \int_{0}^{2\pi} \int_{0}^{\frac{-1+\sqrt{1+4R^2}}{2}} \int_{\sqrt{z}}^{\sqrt{R^2-z^2}} r\, dr\, dz\, d\theta$$

$$= \frac{\pi R^3}{2} - \frac{\pi}{12}\left(1+4R^2\right)\left(-1+\sqrt{1+4R^2}\right)$$

6. The mass of the solid bounded below by $z = \frac{1}{a}\sqrt{x^2+y^2}$ and above by $x^2 + y^2 + z^2 = R^2$, with density $\delta = 3e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}$.

$$\int_{0}^{2\pi} \int_{0}^{\tan^{-1} a} \int_{0}^{R} 3e^{\rho^3} \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\left(1 - \frac{1}{\sqrt{1-a^2}}\right)\left(1 - e^{-R^3}\right)$$

7. The volume inside $x^2 + y^2 + z^2 = 2z$ and below $z = 1 + \sqrt{x^2+y^2}$.

Notice that these are just the sphere $\rho = 1$ and cone $\phi = \frac{\pi}{4}$, shifted up 1 unit. So, we can find the volume inside the standard surfaces in spherical coordinates. or use cylindrical coordinates for the surfaces as stated.

$$\int_{0}^{2\pi} \int_{\frac{\pi}{4}}^{\pi} \int_{0}^{1} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{\sqrt{2z-z^2}} r\, dr\, dz\, d\theta + \int_{0}^{2\pi} \int_{1}^{\frac{2+\sqrt{2}}{2}} \int_{z-1}^{\sqrt{2z-z^2}} r\, dr\, dz\, d\theta$$

$$= \frac{\pi}{3}\left(2+\sqrt{2}\right)$$

8. The center of mass of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = x^2 + y^2$, with density proportional to the distance from the z-axis.

   By symmetry, we know that $\bar{x} = \bar{y} = 0$.

$$\bar{z} = \frac{\int_0^{2\pi} \int_0^1 \int_{r^2}^{r} z \cdot kr \cdot r \, dz \, dr \, d\theta}{\int_0^{2\pi} \int_0^1 \int_{r^2}^{r} kr \cdot r \, dz \, dr \, d\theta} = \frac{4}{7}$$

9. The volume inside the cylinders $x^2 + z^2 = R^2$ and $y^2 + z^2 = R^2$.

$$16 \int_0^R \int_0^x \int_0^{\sqrt{R^2 - x^2}} 1 \, dz \, dy \, dx = \frac{16}{3} R^3$$

10. The volume inside the cylinders $x^2 + z^2 = R^2$, $y^2 + z^2 = R^2$ and $x^2 + y^2 = R^2$.

$$16 \int_0^{\frac{\pi}{4}} \int_0^R \int_0^{\sqrt{R^2 - r^2 \cos^2 \theta}} r \, dz \, dr \, d\theta = 8 \left(2 - \sqrt{2}\right) R^3$$